New class of finite element methods: weak Galerkin methods

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Consider second order elliptic problem:

\[-\nabla \cdot a \nabla u = f, \quad \text{in } \Omega \tag{1}\]
\[u = 0, \quad \text{on } \partial \Omega. \tag{2}\]

Testing (1) by \(v \in H^1_0(\Omega)\) gives

\[-\int_\Omega \nabla \cdot a \nabla uv \, dx = \int_\Omega a \nabla u \cdot \nabla v \, dx - \int_{\partial \Omega} a \nabla u \cdot n v \, ds = \int_\Omega fv \, dx.\]

\[(a \nabla u, \nabla v) = (f, v),\]

where \((f, g) = \int_\Omega fg \, dx\).
PDE and its weak form

PDE: find $u$ satisfies

$$-\nabla \cdot (a \nabla u) = f, \quad \text{in } \Omega$$

$$u = 0, \quad \text{on } \partial \Omega.$$

Its weak form: find $u \in H^1_0(\Omega)$ such that

$$(a \nabla u, \nabla v) = (f, v), \quad \forall v \in H^1_0(\Omega).$$
Infinity vs finite

Weak form: find $u \in H_0^1(\Omega)$ such that

$$(a\nabla u, \nabla v) = (f, v), \quad \forall v \in H_0^1(\Omega).$$

Let $V_h \subset H_0^1(\Omega)$ be a finite dimensional space.

Continuous finite element method: find $u_h \in V_h$ such that

$$(a\nabla u_h, \nabla v_h) = (f, v_h), \quad \forall v_h \in V_h,$$
Continuous finite element method

Find \( u_h \in V_h \) such that

\[
(a \nabla u_h, \nabla v_h) = (f, v_h), \quad \forall v_h \in V_h.
\]

Let \( V_h = \text{Span}\{\phi_1, \cdots, \phi_n\} \) and \( u_h = \sum_{j=1}^{n} c_j \phi_j \), then

\[
\sum_{j=1}^{n} (a \nabla \phi_j, \nabla \phi_i) c_j = (f, \phi_i), \quad i = 1, \cdots, n.
\]

The equation above is a symmetric and positive definite linear system. Solve it to obtain the finite element solution \( u_h \).
Limitations of the continuous finite element methods

- **On approximation functions.** $P_k$ only for triangles and $Q_k$ for quadrilaterals. Hard to construct high order and special elements such as $C^1$ conforming element.

- **On mesh generation.** Only triangular or quadrilateral meshes can be used in 2D. Hybrid meshes or meshes with hanging nodes are not allowed. Not compatible to $hp$ adaptive technique.
**Cause**: Continuity requirement of approximating functions cross element boundaries.

**Solution**: Use discontinuous approximations.
Pros and cons of using discontinuous functions

Pros

- Flexibility on approximation functions. Polynomial $P_k$ can be used on any polygonal element. Easy to construct high order element.
- Flexibility on mesh generation. Hybrid meshes or meshes with hanging nodes are allowed. Compatible to $hp$ adaptive technique.

Cons

- There are more unknowns.
- Complexity in finite element formulations due to enforcing connections of numerical solutions between element boundaries.
Weak Galerkin finite element methods

Weak Galerkin (WG) methods use discontinuous approximations. The WG methods keep the advantages:

- Flexible in approximations. Avoid construction of special elements such as $C^1$ conforming elements.
- Flexible in mesh generation. Hybrid meshes or meshes with hanging nodes can be used.

and minimize the disadvantages:

- Simple formulations.
- Comparable number of unknowns to the continuous finite element methods if implemented appropriately.
Weak functions

Let $T$ be a quadrilateral with $e_j$ for $j = 1, \cdots, 4$ as its four sides. Define

$$v = \begin{cases} v_0 \in P_1(T), & \text{in } T^0 \\ v_b \in P_0(e), & \text{on } e \subset \partial T \end{cases}$$

Define

$$V_h(T) = \{ v \in L^2(T) : v = \{v_0, v_b\} \} = \text{span}\{\phi_1, \cdots, \phi_7\}$$

where

$$\phi_j = \begin{cases} 1, & \text{on } e_i \\ 0, & \text{otherwise} \end{cases}$$

$$j = 1, \cdots, 4$$

$$\phi_5 = \begin{cases} 1, & \text{in } T^0 \\ 0, & \text{on } \partial T \end{cases}$$

$$\phi_6 = \begin{cases} x, & \text{in } T^0 \\ 0, & \text{on } \partial T \end{cases}$$

$$\phi_7 = \begin{cases} y, & \text{in } T^0 \\ 0, & \text{on } \partial T \end{cases}$$
Define a weak gradient $\nabla_w v \in [P_0(T)]^2$ for $v = \{v_0, v_b\} \in V_h(T)$ on the element $T$:

$$(\nabla_w v, q)_T = -(v_0, \nabla \cdot q)_T + (v_b, q \cdot n)_{\partial T}, \quad \forall q \in [P_0(T)]^2.$$ 

Let $\phi_j = \{\phi_{j,0}, \phi_{j,b}\}, j = 1, \cdots, 7$. The definition of the weak gradient gives that for any $q \in [P_0(T)]^2$

$$(\nabla_w \phi_5, q) = -(\phi_{5,0}, \nabla \cdot q)_T + (\phi_{5,b}, q \cdot n)_{\partial T} = 0.$$ 

We have

$$\nabla_w \phi_5 = \nabla_w \phi_6 = \nabla_w \phi_7 = 0.$$ 

Using the definition of $\nabla_w$, we can find for $j = 1, \cdots, 4$

$$\nabla_w \phi_j = \frac{|e_j|}{|T|} n_j.$$ 

Weak gradient $\nabla_w$ for all the basis function $\phi_j$ can be found explicitly.
The local stiffness matrix for the WG method

Denote $Q_b$ the $L^2$ projection to $P_0(e_j)$. $Q_b v_0|_{e_j} = v_0(m_j)$ where $m_j$ is the midpoint of $e_j$.
Define

$$a_T(v, w) = (a \nabla_w v, \nabla_w w)_T + h^{-1} \langle Q_b v_0 - v_b, Q_b w_0 - w_b \rangle_{\partial T}.$$  

The local stiffness matrix $A$ for the WG method on the element $T$ for second order elliptic problem is a $7 \times 7$ matrix

$$A = (a_T(\phi_i, \phi_j)), \quad i, j = 1, \ldots, 7.$$
Weak Galerkin finite element methods

• Define weak function \( v = \{v_0, v_b\} \) such that

\[
v = \begin{cases} v_0, & \text{in } T^0 \\ v_b, & \text{on } \partial T \end{cases}
\]

Define weak Galerkin finite element space

\[
V_h = \{v = \{v_0, v_b\} : v_0|_T \in P_j(T), v_b \in P_\ell(e), e \subset \partial T, v_b = 0 \text{ on } \partial \Omega \}.
\]

• Define a weak gradient \( \nabla_w v \in [P_r(T)]^d \) for \( v \in V_h \) on each element \( T \):

\[
(\nabla_w v, q)_T = -(v_0, \nabla \cdot q)_T + \langle v_b, q \cdot n \rangle_{\partial T}, \quad \forall q \in [P_r(T)]^d.
\]

Weak Galerkin element: \( (P_j(T), P_\ell(e), [P_r(T)]^d) \). For example: \( (P_1(T), P_0(e), [P_0(T)]^d) \).
Weak Galerkin finite element formulation

Define

\[ a(u_h, v_h) = (a \nabla w u_h, \nabla w v_h) + \sum_T h_T^{-1} \langle u_0 - u_b, v_0 - v_b \rangle_{\partial T}. \]

The WG method: find \( u_h = \{u_0, u_b\} \in V_h \) satisfying

\[ a(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_h. \]

Theorem. Let \( u_h \) be the solution of the WG method associated with local spaces \( (P_k(T), P_k(e), [P_{k-1}(T)]^d) \), then

\[ h\|Q_h u - u_h\| + \|Q_h u - u_h\| \leq C h^{k+1} \|u\|_{k+1}, \]

where \( Q_h u \) is the \( L^2 \) projection of \( u \).
Simple formulation: the WG method for the Stokes equations

The weak form of the Stokes equations: find \((u, p) \in [H^1_0(\Omega)]^d \times L^2_0(\Omega)\) that for all \((v, q) \in [H^1_0(\Omega)]^d \times L^2_0(\Omega)\)

\[
(\nabla u, \nabla v) - (\nabla \cdot v, p) = (f, v) \\
(\nabla \cdot u, q) = 0.
\]

The weak Galerkin method: find \((u_h, p_h) \in V_h \times W_h\) such that for all \((v, q) \in V_h \times W_h\),

\[
(\nabla_w u_h, \nabla_w v) + s(u_h, v) - (\nabla_w \cdot v, p_h) = (f, v) \\
(\nabla_w \cdot u_h, q) = 0.
\]
The weak form of the Stokes equations: seeking $u \in H^2_0(\Omega)$ satisfying

$$(\Delta u, \Delta v) = (f, v), \quad \forall v \in H^2_0(\Omega),$$

Weak Galerkin finite element method: seeking $u_h \in V_h$ satisfying

$$(\Delta_w u_h, \Delta_w v) + s(u_h, v) = (f, v), \quad \forall v \in V_h.$$
Implementation of the WG method

The WG method: find $u_h = \{u_0, u_b\} \in V_h$ satisfying

$$a(u_h, v_h) = (f, v_h), \quad \forall v_h = \{v_0, v_b\} \in V_h.$$  

Effective implementation of the WG method:
1. Solve $u_0$ as a function of $u_b$ from the following local system on element $T$,

$$a(u_h, v_h) = (f, v_h), \quad \forall v_h = \{v_0, 0\} \in V_h. \tag{3}$$

2. Solve $u_b$ from the following global system,

$$a(u_h, v_h) = (f, v_h), \quad \forall v_h = \{0, v_b\} \in V_h. \tag{4}$$

**Theorem.** The global system (4) is symmetric and positive definite.

For the lowest order WG method, the number of unknowns of (4) is

$$\# \text{ of unknowns} = \# \text{ of interior edges}.$$
Consider the model problem,

$$\begin{align*}
-\nabla \cdot a \nabla u & = f, \quad \text{in } \Omega \\
u & = 0, \quad \text{on } \partial \Omega.
\end{align*}$$

Rewrite the problem as the system of first order equations,

$$\begin{align*}
q + a \nabla u & = 0, \quad \text{in } \Omega, \\
\nabla \cdot q & = f, \quad \text{in } \Omega, \\
u & = 0, \quad \text{on } \partial \Omega.
\end{align*}$$

The least-squares method: find $(q, u) \in H(div; \Omega) \times H^1_0(\Omega)$ such that for any $(\sigma, v) \in H(div; \Omega) \times H^1_0(\Omega)$,

$$\begin{align*}
(q + a \nabla u, \sigma + a \nabla v) + (\nabla \cdot q, \nabla \cdot \sigma) & = (f, \nabla \cdot \sigma).
\end{align*}$$
The WG Least-squares method

The least-squares method: find \((\mathbf{q}, \mathbf{u}) \in H(\text{div}; \Omega) \times H^1_0(\Omega)\) such that for any \((\mathbf{\sigma}, \mathbf{v}) \in H(\text{div}; \Omega) \times H^1_0(\Omega)\),

\[
(\mathbf{q} + a \nabla \mathbf{u}, \mathbf{\sigma} + a \nabla \mathbf{v}) + (\nabla \cdot \mathbf{q}, \nabla \cdot \mathbf{\sigma}) = (f, \nabla \cdot \mathbf{\sigma}).
\]

The WG least-squares method: find \((\mathbf{q}_h, \mathbf{u}_h) \in \Sigma_h \times V_h\) such that for any \((\mathbf{\sigma}, \mathbf{v}) \in \Sigma_h \times V_h\),

\[
(\mathbf{q}_h + a \nabla_w \mathbf{u}_h, \mathbf{\sigma} + a \nabla_w \mathbf{v}) + (\nabla_w \cdot \mathbf{q}_h, \nabla_w \cdot \mathbf{\sigma}) + s_1(\mathbf{u}_h, \mathbf{v}) + s_2(\mathbf{q}_h, \mathbf{\sigma}) = (f, \nabla \cdot \mathbf{\sigma}).
\]
Define
\[ \mathcal{D}_h = \{ n_e : n_e \text{ is unit and normal to } e, \ e \in \mathcal{E}_h \}, \]
\[ V_h = \{ v = \{ v_0, v_b \} : v_0|_T \in P_{k+1}(T), v_b|_e \in P_k(e), e \in \partial T, v_b = 0, \text{ on } \partial \Omega \}, \]
\[ \Sigma_h = \{ \sigma = \{ \sigma_0, \sigma_b \} : \sigma_0|_T \in [P_k(T)]^d, \sigma_b|_e = \sigma_b n_e, \sigma_b|_e \in P_k(e), e \in \partial T \}. \]

Define
\[ s_1(w, v) = \sum_{T \in \mathcal{T}_h} h^{-1} \langle Q_b w_0 - w_b, Q_b v_0 - v_b \rangle_{\partial T}, \]
\[ s_2(t, \sigma) = \sum_{T \in \mathcal{T}_h} h \langle (t_0 - t_b) \cdot n, (\sigma_0 - \sigma_b) \cdot n \rangle_{\partial T}, \]

The WG least-squares method: find \((q_h, u_h) \in \Sigma_h \times V_h\) such that for any \((\sigma, v) \in \Sigma_h \times V_h\),
\[ (q_h + a \nabla w u_h, \sigma + a \nabla w v) + (\nabla w \cdot q_h, \nabla w \cdot \sigma) + s_1(u_h, v) + s_2(q_h, \sigma) = (f, \nabla \cdot \sigma). \]
We introduce a norm $\| \cdot \|_V$ in $V_h$ as

$$\|v\|_V^2 = \sum_{T \in \mathcal{T}_h} \|\nabla_w v\|_T^2 + s_1(v,v),$$

and a norm $\| \cdot \|_\Sigma$ in $\Sigma_h$ as

$$\|\sigma\|_\Sigma^2 = \sum_{T \in \mathcal{T}_h} \|\nabla_w \sigma\|_T^2 + \|\sigma_0\|^2 + s_2(\sigma,\sigma).$$

**Lemma.** There is a constant $C$ such that for all $(\sigma, v) \in \Sigma_h \times V_h$

$$C(\|\sigma\|_\Sigma^2 + \|v\|_V^2) \leq a(v,\sigma; v,\sigma).$$

**Theorem.** Assume the exact solution $u \in H^{k+2}(\Omega)$ and $q \in [H^{k+1}(\Omega)]^d$. Then, there exists a constant $C$ such that

$$\|u_h - Q_h u\|_V + \|q_h - Q_h q\|_\Sigma \leq Ch^{k+1}(\|u\|_{k+2} + \|q\|_{k+1}).$$
Implementation of the WG least-squares method

The WG least-squares method: find \((q_h, u_h) \in \Sigma_h \times V_h\) such that for any \((\sigma, v) \in \Sigma_h \times V_h\),

\[
(q_h + a \nabla_w u_h, \sigma + a \nabla_w v) + (\nabla_w \cdot q_h, \nabla_w \cdot \sigma) + s_1(u_h, v) + s_2(q_h, \sigma) = (f, \nabla \cdot \sigma).
\]

Effective implementation of the WG least-squares method:
1. Solve the local systems on each element \(T \in \mathcal{T}_h\) for any \(v = \{v_0, 0\} \in V_h(T)\) and \(\sigma = \{\sigma_0, 0\} \in \Sigma_h(T)\),

\[
a(u_h, q_h; v, \sigma) = (f, \nabla_w \cdot \sigma)_T.
\]

2. Solve a global system,

\[
a(u_h, q_h; v, \sigma) = 0, \quad \forall v = \{0, v_b\} \in V_h, \sigma = \{0, q_b\} \in \Sigma_h.
\]

For the WG least-squares method with \(k = 0\),

\[
\text{\# of unknowns} = 2 \times \text{\# of interior edges}.
\]
Example: Let $\Omega = (0, 1) \times (0, 1)$ and the exact solution is given by

$$u = x(1 - x)y(1 - y).$$
The weak Galerkin method for the elliptic interface problems

Consider a elliptic interface problem,

\[-\nabla \cdot A\nabla u = f, \quad \text{in } \Omega,\]
\[u = 0, \quad \text{on } \partial \Omega \setminus \Gamma,\]
\[[u]_\Gamma = \psi, \quad \text{on } \Gamma,\]
\[[A\nabla u \cdot n]_\Gamma = \phi, \quad \text{on } \Gamma,\]

\[V_h = \{ v = \{ v_0, v_b \} : v_0|_T \in P_k(T), v_b|_e \in P_k(e), e \in \partial T, v_b = 0, \text{on } \partial \Omega \}\]

The weak Galerkin method: find \( u_h \in V_h \) such that

\[
(A\nabla_w u_h, \nabla_w v) + \sum_T h^{-1} \left\langle u_0 - u_b, v_0 - v_b \right\rangle_{\partial T} = (f, v_0)
\]

\[+ \left\langle \psi, A\nabla_w v \cdot n \right\rangle_{\Gamma} - \left\langle \phi, v_b \right\rangle_{\Gamma} - \left\langle \psi, v_0 - v_b \right\rangle_{\Gamma}, \forall v \in V_h.\]
Elliptic interface problems: Example 1

\[ \Omega = (0, 1)^2 \text{ with } \Omega_1 = [0.2, 0.8]^2 \text{ and } \Omega_2 = \Omega / \Omega_1. \]

Then the exact solution:

\[
\begin{align*}
    u &= \begin{cases} 
    5 + 5(x^2 + y^2), & \text{if } (x, y) \in \Omega_1 \\
    x^2 + y^2 + \sin(x + y), & \text{if } (x, y) \in \Omega_2
    \end{cases}
\end{align*}
\]

Permeability:

\[
\begin{align*}
    A &= \begin{cases} 
    1, & \text{if } (x, y) \in \Omega_1 \\
    2 + \sin(x + y), & \text{if } (x, y) \in \Omega_2
    \end{cases}
\end{align*}
\]
Mesh 1

Mesh 2

Solution on mesh 1

Solution on mesh 2
The exact solution is

\[ u(x, y) = \begin{cases} 
  x - y^2 + 10 & \text{if} \ (x, y) \in \Omega_1 \\
  e^x \cos \pi y & \text{otherwise}
\end{cases} \]

**Figure**: The interface \( \Gamma \) in Example 2.
<table>
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<th>Solution</th>
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**Figure**: The WG approximation of Example 2 on mesh level 5. Left: Numerical solution; Right: Exact solution.
• The weak Galerkin finite element methods represent advanced methodology for handling discontinuous functions in finite element procedure.
• The weak Galerkin finite element methods have the flexibility of using discontinuous elements and the simplicity of using continuous elements.