STATIONARY AND ERGODIC HYPERRANDOM FUNCTIONS

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A new mathematical apparatus is developed for description of hyperrandom functions (a special type of indeterminate functions, for which the probabilistic measure is not defined). The methods for description of special classes of hyperrandom functions, possessing the stationary and ergodic properties, are also developed. The potentialities of application of these methods in radio engineering are illustrated by a concrete example.

The surrounding world is extremely complex and dynamic. Because of a great number of rapidly varying linkages, ambiguity or absence of some data, we hardly could describe our world only with the use of determinate methods. Particularly, we often have to resort to indeterminate descriptions when investigating complex radio-engineering processes and systems, whose characteristics and behavior depend on many different factors. For a long time, the probability theory and mathematical statistic were the only alternatives to the determinate approach. Indeed, the probabilistic and statistical methods permit to cope with many problems. However, there are situations, when application of these methods becomes ineffective for some or other reasons. In this connection, new mathematical theories, proposed for describing indeterminate phenomena, began to appear: the theory of fuzzy sets [1–2], already well elaborated and occupying its niche; the intensively developing theory of determinate chaos [3–6]; and a quite new theory of hyperrandom phenomena, which is now in the making [7–10].

It is well known [11–13] that the random phenomena can be defined with the aid of the probabilistic space set by a triple \((\Omega, \mathcal{F}, P)\), where \(\Omega\) is the space of elementary events \(\omega \in \Omega\), \(\mathcal{F}\) is Borel's field (\(\sigma\)-algebra of subsets of events), and \(P\) is the probabilistic measure of subsets of events.

The hyperrandom phenomena are characterized with a quadruple \((\Omega, \mathcal{G}, G, P_g)\) [8], where \(\Omega\) is the space of elementary events \(\omega \in \Omega\), \(G\) is Borel’s field (\(\sigma\)-algebra of subsets of events), \(G\) is the totality of conditions \(g \in G\), and \(P_g\) is the probabilistic measure of subsets of events depending on the condition \(g\). Thus, the probabilistic measure is set for all subsets of events and for all conditions \(g \in G\) defining the occurrence of the events. At the same time, the measure for the conditions \(g \in G\) remains indeterminate. This approach permits to introduce in the models of real events some elements of indeterminacy of a higher level than the random type indeterminacy.

We cannot set the probabilistic measure for a hyperrandom event, but we can establish one-to-one correspondence for the quantities [8], which characterize quantitatively the range of variation of probability of an event: the upper and the lower probability boundaries defined, respectively, for an event \(A\) in the form

\[
P_g(A) = \sup_{g \in G} P(A | g), P_e(A) = \inf_{g \in G} P(A | g).
\]

Based on this approach, in [8, 9] we suggested a mathematical apparatus for description of hyperrandom quantities and functions. The basis for this approach refers to the distribution function boundaries \(F_g(x)\) and \(F_e(x)\) and to their derivatives (the boundaries of distribution density and the boundaries of the characteristic function), and includes a number of auxiliary characteristics: the mean values of the boundaries, variances of boundaries, correlation and covariance.

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functions of boundaries, etc. The set of these characteristics makes it possible to describe and compare different hyperrandom events. However, in order to utilize this apparatus effectively, we have to deal with a great amount of data, which is usually inaccessible in the case of practical problems. In this connection, in [10] we developed another approach, based on the boundaries of moments, which can be calculated rather easily.

The purpose of this paper is comparison of both approaches as applied to hyperrandom functions possessing specific properties of stationarity and ergodicity, and illustration of application of the obtained results to radio-engineering tasks.

The stationary hyperrandom functions. A hyperrandom function $X(t) = \{X(t, g), g \in G\}$, where $X(t, g)$ is a random function under a condition $g$, will be called stationary in the narrow (strict) sense, if the boundaries of its $L$-dimensional distributions at any $L$ depend only on duration of the intervals $t_2 - t_1, ..., t_{n-1} - t_1$, and are invariant to position of these intervals on the $t$-axis. Hyperrandom functions not belonging to these functions will be called non-stationary in the narrow sense.

The properties of a stationary hyperrandom function are similar to those of a stationary random function: the boundaries of its many-dimensional distribution function, the boundaries of its many-dimensional distribution density, and the boundaries of its many-dimensional characteristic function do not depend on position on the $t$-axis. In addition, the above single-dimensional characteristics do not depend on the argument $t$, i.e., $f_{X(t)}(x_1; t) = f_{X(t)}(x_1; t)$, $f_{X(t)}(x_1; t) = f_{X(t)}(x_1; t)$, $f_{X(t)}(x_1; x_2; t_1, t_2) = f_{X(t)}(x_1; x_2; t_2)$, $f_{X(t)}(x_1; x_2; t_1, t_2) = f_{X(t)}(x_1; x_2; t_2)$.

The moment functions of boundaries of a stationary hyperrandom function $X(t)$ possess the following properties: the mean values of boundaries and variances of boundaries are constant $(m_{X(t)}(t) = m_{X(t)}$, $m_{X(t)}(t) = m_{X(t)}$, $D_{X(t)}(t) = D_{X(t)}$, $D_{X(t)}(t) = D_{X(t)}$), while the covariance functions of boundaries

$$K_{X(t)}(t_1, t_2) = M_{X(t)}[X(t_1)X(t_2)]$$

the correlation functions of the boundaries

$$R_{X(t)}(t_1, t_2) = M_{X(t)}[X(t_1) - m_{X(t)}][X(t_2) - m_{X(t)}]$$

and the normalized correlation functions of the boundaries

$$\rho_{X(t)}(t_1, t_2) = \frac{R_{X(t)}(t_1, t_2)}{D_{X(t)}(t_1)D_{X(t)}(t_2)}$$

do not depend on position of the interval $\tau = t_2 - t_1$ on the $t$-axis:

$$K_{X(t)}(t_1, t_2) = K_{X(t)}(\tau)$$

$K_{X(t)}(t_1, t_2) = K_{X(t)}(\tau)$

$R_{X(t)}(t_1, t_2) = R_{X(t)}(\tau)$

$$\rho_{X(t)}(t_1, t_2) = \rho_{X(t)}(\tau) = R_{X(t)}(\tau)/D_{X(t)}$$

A hyperrandom function $X(t)$ will be called stationary in the broad sense, if the mean values of its boundaries are constant $(m_{X(t)}(t) = m_{X(t)}$, $m_{X(t)}(t) = m_{X(t)}$, $D_{X(t)}(t) = D_{X(t)}$, $D_{X(t)}(t) = D_{X(t)}$), and the covariance functions of the boundaries depend only on the difference between the arguments $t$-values:

$$K_{X(t)}(t_1, t_2) = M_{X(t)}[X(t_1)X(t_2)] = K_{X(t)}(\tau)$$

$K_{X(t)}(t_1, t_2) = M_{X(t)}[X(t_1)X(t_2)] = K_{X(t)}(\tau)$

The hyperrandom functions, stationary in the narrow sense, are also stationary in the broad sense. The reverse assertion is not true in the general case.

Two hyperrandom functions $X(t)$ and $Y(t)$ will be called simultaneously steady linked in the broad sense if the mean values of their boundaries are constant, and their mutual covariance functions of boundaries are invariant to displacement on the $t$-axis:

$$K_{X(t)}(t_1, t_2) = M_{X(t)}[X(t_1)Y(t_2)] = K_{X(t)}(\tau)$$

$K_{X(t)}(t_1, t_2) = M_{X(t)}[X(t_1)Y(t_2)] = K_{X(t)}(\tau)$

$K_{Y(t)}(t_1, t_2) = M_{Y(t)}[X(t_1)Y(t_2)] = K_{Y(t)}(\tau)$

$K_{Y(t)}(t_1, t_2) = M_{Y(t)}[X(t_1)Y(t_2)] = K_{Y(t)}(\tau)$

Note that stationarity of hyperrandom functions in the broad sense does not guarantee their simultaneous stationary linkage in the broad sense.

The correlation functions of boundaries and normalized correlation functions of boundaries of real-valued stationary hyperrandom functions $X(t)$ and $Y(t)$ possess the following properties:

$$40$$
1) $|R_{xx}(\tau)| \leq D_{xx}, \quad |\rho_{xx}(\tau)| \leq 1$, $|R_{xy}(\tau)| \leq D_{xy}, \quad |\rho_{xy}(\tau)| \leq 1$

2) The maximum values of correlation functions of boundaries and of normalized correlation functions of boundaries of a hyperbroad function take place at $\tau = 0$.

3) The functions $R_{xy}(\tau), R_{xx}(\tau), \rho_{xy}(\tau), \rho_{xx}(\tau)$ are even.

4) $R_{xy}(\tau) = R_{xy}(-\tau), \quad R_{xx}(\tau) = R_{xx}(-\tau), \quad \rho_{xy}(\tau) = \rho_{xy}(-\tau), \quad \rho_{xx}(\tau) = \rho_{xx}(-\tau)$, where $R_{xy}(\tau), R_{xx}(\tau)$ are cross-correlation functions of boundaries, $\rho_{xy}(\tau), \rho_{xx}(\tau)$ are normalized cross-correlation functions of boundaries:

$$
\rho_{xy}(\tau) = R_{xy}(\tau)/D_{xy}, \quad \rho_{xx}(\tau) = R_{xx}(\tau)/D_{xx}, \quad D_{xy} = R_{xy}(0)
$$

A hyperbroad function $X(t) = \{X(t|g), g \in G\}$ will be called stationary in the narrow sense under all conditions $g \in G$ if for all $g$ its conditional $L$-dimensional distributions at any $L$ stay only on duration of the intervals $t_2 - t_1, ..., t_L - t_1$ and are invariant to translation of these intervals on the $t$-axis.

A hyperbroad function $X(t)$ will be called stationary in the broad sense under all conditions $g \in G$ if for any fixed condition $g$ its conditional mean value $m_{x|g}(t) = \int f(x; t|g) dx$ does not depend on the argument $t (m_{x|g}(t) = m_{x|g})$, and the conditional covariance function

$$
K_{x|g}(t_1, t_2) = \int \int f(x_1, x_2; t_1, t_2|g) dx_1 dx_2
$$

depends only on the difference between the values of the argument $t$ and on the condition $g$: $K_{x|g}(t_1, t_2) = K_{x|g}(t)$.

Note that in this case the conditional correlation function

$$
R_{x|g}(t_1, t_2) = \int \int (x_1 - m_{x|g})(x_2 - m_{x|g}) f(x_1, x_2; t_1, t_2|g) dx_1 dx_2
$$

also depends only on $t$ and $g$.

It can be easily seen that the boundaries of the mean value $m_{x}(t) = \sup_{g \in G} m_{x|g}(t), m_{x}(t) = \inf_{g \in G} m_{x|g}(t)$ of the stationary in the broad sense and under all conditions $g$ hyperbroad function do not depend on time $t$, i.e., $m_{x}(t) = m_{x}, m_{x}(t) = m_{x}$, while the boundaries of the covariance function $K_{xx}(\tau) = \sup_{g \in G} K_{x|g}(\tau), K_{xx}(\tau) = \inf_{g \in G} K_{x|g}(\tau)$, and the boundaries of the correlation function $R_{xx}(\tau) = \sup_{g \in G} R_{x|g}(\tau), R_{xx}(\tau) = \inf_{g \in G} R_{x|g}(\tau)$ depend on $\tau$.

The hyperbroad functions $X(t)$ and $Y(t)$ will be called simultaneously steadily linked under all conditions $g$ if the conditional mean values $m_{x|g}(t)$ and $m_{y|g}(t)$ of these functions do not depend on the argument $t (m_{x|g}(t) = m_{x|g}, m_{y|g}(t) = m_{y|g})$, and the conditional mutual covariance function

$$
K_{x'y'|g}(t_1, t_2) = \int \int f(x, y; t_1, t_2|g) dx dy
$$

is invariant to displacement on the $t$-axis: $K_{x'y'|g}(t_1, t_2) = K_{x'y'|g}(0)$. Then the conditional cross-correlation function

$$
R_{x'y'|g}(t_1, t_2) = \int \int (x - m_{x|g})(y - m_{y|g}) f(x, y; t_1, t_2|g) dx dy
$$

is also invariant to displacement along the $t$-axis: $R_{xy}(t_1, t_2) = R_{x'y'|g}(t)$. It can be easily seen that the boundaries of the mutually covariance function $K_{x'y'|g}(\tau) = \sup_{g \in G} K_{x'y'|g}(\tau), K_{x'y'|g}(\tau) = \inf_{g \in G} K_{x'y'|g}(\tau)$, and boundaries of the cross-correlation function $R_{xy}(\tau) = \sup_{g \in G} R_{x'y'|g}(\tau), R_{xy}(\tau) = \inf_{g \in G} R_{x'y'|g}(\tau)$ depend only on $\tau$.
It should be stressed that the concepts of a stationary in the broad sense hyper-random function and of a function, stationary in the broad sense under all conditions, differ from each other. Their common point is infinity of realization and invariance to a shift of certain (but differing from one another) characteristics.

**Spectral description of hyper-random functions.** In a number of cases, spectral representation of hyper-random functions facilitates their analysis substantially. First of all, this refers to the functions possessing the property of stationarity.

Introduce the term of power spectral densities of the upper and lower boundaries (the energy spectra of boundaries) of a stationary hyper-random function \( X(t) \) as applied to the functions \( S_{\text{sys}}(f) \) and \( S_{\text{sys}}(f) \), linked with the covariance functions of boundaries \( K_{\text{sys}}(f) \) and \( K_{\text{sys}}(f) \) via following relations:

\[
S_{\text{sys}}(f) = \int_{-\infty}^{\infty} K_{\text{sys}}(\tau) \exp(-j2\pi ft)\,d\tau, \quad S_{\text{sys}}(f) = \int_{-\infty}^{\infty} K_{\text{sys}}(\tau) \exp(-j2\pi ft)\,d\tau
\]

\[
K_{\text{sys}}(\tau) = \int_{-\infty}^{\infty} S_{\text{sys}}(f) \exp(j2\pi ft)\,df, \quad K_{\text{sys}}(\tau) = \int_{-\infty}^{\infty} S_{\text{sys}}(f) \exp(j2\pi ft)\,df
\]

The power spectral densities of boundaries possess the properties inherent in the power spectral density of a random process: the energy spectra of boundaries both for a real and an imaginary function \( X(t) \) are real-valued and nonnegative, i.e. \( S_{\text{sys}}(f) \geq 0 \) and \( S_{\text{sys}}(f) \geq 0 \); the power spectral densities of boundaries of a real hyper-random function \( X(t) \) are even, i.e. \( S_{\text{sys}}(f) = S_{\text{sys}}(-f) \) and \( S_{\text{sys}}(f) = S_{\text{sys}}(-f) \) (this stems from the fact that the covariance functions of boundaries of stationary hyper-random functions are even).

By the hyper-random white noise \( N(t) \) is meant a stationary hyper-random function \( N(t) \) with zero mean values of its boundaries, where the power spectral densities of the boundaries represent some constant quantities, i.e. \( S_{\text{sys}} = N_f / 2 \) and \( S_{\text{sys}} = N_f / 2 \), where \( N_f \) and \( N_f \) are constant.

It can be easily seen that the covariance functions of boundaries of the hyper-random white noise represent \( \delta \)-functions: \( K_{\text{sys}}(\tau) = N_f \delta(\tau) / 2 \) and \( K_{\text{sys}}(\tau) = N_f \delta(\tau) / 2 \). Note that the same expression is used for describing the correlation functions of boundaries of the hyper-random white noise.

It should be stressed that when defining a hyper-random white noise, as well as a random white noise, we do not employ the concepts of Gaussian nature and independence of cuts. It means that the hyper-random white noise may be non-Gaussian and with dependent (in conformity to the theory of hyper-random events [9]) cuts.

The method of spectral description of hyper-random functions permits extension to the case of steadily linked hyper-random functions.

By mutual power spectral densities of boundaries of two steadily linked hyper-random functions \( X(t) \) and \( Y(t) \) are meant the determinate functions \( S_{\text{sys}}(f) \) and \( S_{\text{sys}}(f) \) defined as the Fourier transform of mutual covariance functions of boundaries \( K_{\text{sys}}(\tau) \) and \( K_{\text{sys}}(\tau) \):

\[
S_{\text{sys}}(f) = \int_{-\infty}^{\infty} K_{\text{sys}}(\tau) \exp(-j2\pi ft)\,d\tau, \quad S_{\text{sys}}(f) = \int_{-\infty}^{\infty} K_{\text{sys}}(\tau) \exp(-j2\pi ft)\,d\tau
\]

\[
K_{\text{sys}}(\tau) = \int_{-\infty}^{\infty} S_{\text{sys}}(f) \exp(j2\pi ft)\,df, \quad K_{\text{sys}}(\tau) = \int_{-\infty}^{\infty} S_{\text{sys}}(f) \exp(j2\pi ft)\,df
\]

The mutual covariance functions of boundaries are related to the mutual power spectral densities of boundaries through the backward Fourier transform:

\[
K_{\text{sys}}(\tau) = \int_{-\infty}^{\infty} S_{\text{sys}}(f) \exp(j2\pi ft)\,df, \quad K_{\text{sys}}(\tau) = \int_{-\infty}^{\infty} S_{\text{sys}}(f) \exp(j2\pi ft)\,df
\]

As distinct from power spectral densities of boundaries of a single hyper-random function, the mutual power spectral densities of boundaries \( S_{\text{sys}}(f) \) and \( S_{\text{sys}}(f) \) are not real-valued functions in the general case. Moreover, they are not even, but possess the property of Hermitian conjugation: \( S_{\text{sys}}(f) = S_{\text{sys}}^*(f) \), \( S_{\text{sys}}(f) = S_{\text{sys}}^*(f) \).
It can be easily seen that the mutual power spectral densities of boundaries $\tilde{S}_{xy}(f)$ and $\tilde{S}_{yx}(f)$ of the functions $X(t)$ and $Y(t)$ are related to power spectral densities of boundaries $S_{XX}(f), S_{YY}(f)$ and $S_{XY}(f), S_{YX}(f)$ through the following inequalities:

$$\begin{align*}
|\tilde{S}_{xy}(f)|^2 &\leq S_{XX}(f) S_{yy}(f), \\
|\tilde{S}_{yx}(f)|^2 &\leq S_{YY}(f) S_{yx}(f).
\end{align*}$$

In order to characterize the rate and type of relations between the hyprandom functions $X(t)$ and $Y(t)$, we can use the functions of frequency coherency of boundaries $\gamma_{xy}^2(f)$ and $\gamma_{yx}^2(f)$ defined similar to a function of frequency coherency of two random functions:

$$\begin{align*}
\gamma_{xy}^2(f) &= \frac{|\tilde{S}_{xy}(f)|^2}{S_{XX}(f) S_{yy}(f)}, \\
\gamma_{yx}^2(f) &= \frac{|\tilde{S}_{yx}(f)|^2}{S_{YY}(f) S_{yx}(f)}.
\end{align*}$$

The functions of frequency coherency of boundaries are confined within the interval $[0, 1]$. If the functions $X(t)$ and $Y(t)$ are uncorrelated, then, for all $f \neq 0$, $\gamma_{xy}^2(f) = \gamma_{yx}^2(f) = 0$; if they are linearly dependent, then $\gamma_{xy}^2(f) = \gamma_{yx}^2(f) = 1$.

The functions of frequency coherency of boundaries are similar to normalized correlation functions of boundaries $r_{xy}(\tau)$ and $r_{yx}(\tau)$ but, as distinct from the latter, they characterize not only linear but also nonlinear linkages between hyprandom functions.

By instantaneous spectrum of a hyprandom function $X(t) = \{X(t \mid g), g \in G\}$ under the condition $g$ is meant a complex hyprandom function $\hat{S}_{x/g}(f)$ which is linked with the observed under condition $g$ process $X(t \mid g)$ through the Fourier transform:

$$\hat{S}_{x/g}(f) = \int_{-\infty}^{\infty} \{X(t \mid g) \exp(-j2\pi ft)\} dt.$$

The instantaneous spectrum of a stationary under all conditions $g$ hyprandom function possesses the properties, similar to the properties of the instantaneous spectrum of a random function stationary in the broad sense. Particularly, the conditional mean value $m_{x/g}(f)$ of the instantaneous spectrum of the hyprandom function $X(t)$ is linked with the conditional mean value $m_{x/g}(f)$ of the function $X(t)$ through expressions $m_{x/g}(f) = m_{x/g} \delta(f)$.

Define the conditional power spectrum $S_{x/g}(f)$ of the function $X(t)$ as the Fourier transform of the conditional covariance function $K_{x/g}(\tau)$:

$$S_{x/g}(f) = \int_{-\infty}^{\infty} K_{x/g}(\tau) \exp(-j2\pi ft) d\tau,$$

where $K_{x/g}(\tau)$ is related to $S_{x/g}(f)$ via the backward Fourier transform:

$$K_{x/g}(\tau) = \int_{-\infty}^{\infty} S_{x/g}(f) \exp(j2\pi ft) df.$$

It can be easily shown that the conditional covariance function of the instantaneous spectrum $K_{x/g}(f_1, f_2)$ of the stationary under all conditions hyprandom function $X(t)$ can be represented in the form

$$K_{x/g}(f_1, f_2) = S_{x/g}(f_1) \delta(f_2 - f_1).$$

It follows from (1) that the instantaneous spectrum of a stationary hyprandom function is not a stationary function; the samples of the instantaneous spectrum, corresponding to different frequencies, are orthogonal; at the zero mean values of boundaries the samples of the instantaneous spectrum, corresponding to different frequencies, are not only orthogonal but also uncorrelated.

Note that the conditional power spectrum $S_{x/g}(f)$ is linked with the conditional instantaneous spectrum $\hat{S}_{x/g}(f)$ set on the interval $T$, via the following relation:
\[ S_{\text{s}E}(f) = \lim_{T \to \infty} \frac{1}{T} \mathcal{M}[S_{\text{st}}(f)S_{\text{st}}^*(f)] \]

where \( \mathcal{M}[\cdot] \) is the mean value operator.

The boundaries of the energy spectrum can be represented as

\[ S_{\text{ax}}(f) = \sup_{g \in \mathcal{G}} S_{\text{s}E}(f), \quad S_{\text{lx}}(f) = \inf_{g \in \mathcal{G}} S_{\text{s}E}(f) \]

It can be easily seen that the boundaries of the energy spectrum of a stationary hyperrandom function are linked with its instantaneous spectrum under condition \( g \) by the relationships

\[ S_{\text{ax}}(f) = \lim_{T \to \infty} \sup_{g \in \mathcal{G}} \frac{1}{T} \mathcal{M}[S_{\text{st}}(f)S_{\text{st}}^*(f)] \]
\[ S_{\text{lx}}(f) = \lim_{T \to \infty} \inf_{g \in \mathcal{G}} \frac{1}{T} \mathcal{M}[S_{\text{st}}(f)S_{\text{st}}^*(f)] \]

Assign the name of hyperrandom white noise under all conditions to a stationary under all conditions hyperrandom function \( \mathcal{H}(t) \) whose conditional mean value is zero, and the conditional power spectrum is invariant with frequency, i.e.,
\[ S_{\text{s}E}(t) = N_g \delta(t) / 2 \]

where \( N_g \) is a constant value depending in the general case on the condition \( g \).

The conditional covariance function of such noise represents the \( \delta \)-function: \( K_{\text{xy}}(\tau) = N_g \delta(\tau)/2 \). The same expression is used for describing its correlation function. Note that the hyperrandom white noise under all conditions may be non-Gaussian.

Define the conditional mutual power spectrum \( \hat{S}_{\text{xy}}(f) \) of the stationary under all conditions hyperrandom functions \( X(t) \) and \( Y(t) \) as the Fourier transform of the conditional mutual covariance function \( K_{\text{xy}}(\tau) \):

\[ \hat{S}_{\text{xy}}(f) = \int_{-\infty}^{\infty} K_{\text{xy}}(\tau) \exp(-j2\pi f \tau) \, d\tau \]

where \( K_{\text{xy}}(\tau) \) is related to \( \hat{S}_{\text{xy}}(f) \) through the backward Fourier transform:

\[ K_{\text{xy}}(\tau) = \int_{-\infty}^{\infty} \hat{S}_{\text{xy}}(f) \exp(j2\pi f \tau) \, df \]

The boundaries of the mutual energy spectrum can be defined as follows:

\[ \hat{S}_{\text{xx}}(f) = \sup_{g \in \mathcal{G}} \hat{S}_{\text{xy}}(f), \quad \hat{S}_{\text{yy}}(f) = \inf_{g \in \mathcal{G}} \hat{S}_{\text{xy}}(f) \]

One should bear in mind that the conditional mutual power spectrum \( \hat{S}_{\text{xy}}(f) \) and the boundaries of the mutual energy spectrum \( \hat{S}_{\text{xx}}(f) \) and \( \hat{S}_{\text{yy}}(f) \) in the general case are not real functions, not even, and possess the property of Hermitian conjugation: \( \hat{S}_{\text{xy}}(f) = \hat{S}_{\text{xy}}^*(f) \)

In order to characterize the rate and type of relations between hyperrandom functions \( X(t) \) and \( Y(t) \), we can use the boundaries of the function of frequency coherence of boundaries \( \gamma_{\text{xy}}^2(f) \) and \( \gamma_{\text{yy}}^2(f) \) defined as

\[ \gamma_{\text{xy}}^2(f) = \frac{|\hat{S}_{\text{xy}}(f)|^2}{\hat{S}_{\text{xx}}(f)\hat{S}_{\text{yy}}(f)} \]
\[ \gamma_{\text{yy}}^2(f) = \frac{|\hat{S}_{\text{yy}}(f)|^2}{\hat{S}_{\text{xx}}(f)\hat{S}_{\text{yy}}(f)} \]

Note that the conditional mutual power spectrum \( \hat{S}_{\text{xy}}(f) \) is related to conditional mutual power spectra \( S_{\text{s}E}(f) \) and \( S_{\text{xy}}(f) \) through the following inequalities:

\[ |\hat{S}_{\text{xy}}(f)|^2 \leq S_{\text{s}E}(f)S_{\text{xy}}(f) \]

However, the boundaries of the mutual spectral power density \( \hat{S}_{\text{xy}}(f) \) and \( \hat{S}_{\text{yy}}(f) \) have no such link with the boundaries of spectral power densities \( S_{\text{ax}}(f) \) and \( S_{\text{yy}}(f) \) \( \hat{S}_{\text{xy}}(f) \) and \( \hat{S}_{\text{yy}}(f) \). i.e., the inequalities \( |\hat{S}_{\text{xy}}(f)|^2 \leq S_{\text{ax}}(f)S_{\text{yy}}(f) \)

|\( \hat{S}_{\text{yy}}(f) |^2 \leq S_{\text{xx}}(f)S_{\text{yy}}(f) \) are not always true. Because of this, the boundaries of the function of frequency coherence \( \gamma_{\text{xy}}^2(f) \) \( \gamma_{\text{yy}}^2(f) \) may take values exceeding unity.

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The ergodic hyperrandom functions. Some hyperrandom functions possess a specific property of ergodicity. Consider a hyperrandom function \( X(t) \) permitting decomposition into separate random functions, defined on non-intersecting intervals \( T_g \) of duration \( T \), where the conditions \( g \) remain unchanged \( g = 0, \pm 1, \pm 2, \ldots \).

Let \( X_g(t) \) be fragments of the function \( X(t) \), corresponding to the intervals \( T_g \) and reduced to the interval \([-T/2, T/2] \):

\[
X_g(t - T(g + 0.5)) = \begin{cases} 
X(t), & \text{if } t \in T_g, \\
0, & \text{if } t \notin T_g.
\end{cases}
\]

The function \( X_g(t) \) at fixed conditions represents the random function of the argument \( t \in [-T/2, T/2] \). The totality of all these functions, at indeterminate conditions \( g = 0, \pm 1, \pm 2, \ldots \) forms the hyperrandom function \( Y(t) = X_g(t) \), \( g = 0, \pm 1, \ldots \). Assume that within the considered interval \( T \) the hyperrandom function \( Y(t) \) possesses the property of stationarity for all conditions \( g \).

Any function \( \varphi(Y(t_1), \ldots, Y(t_L)) \) of the set of values of the hyperrandom function \( Y(t) \) in fixed points \( t_1, \ldots, t_L \in [-T/2, T/2] \) is a hyperrandom quantity. If this function \( \varphi(Y(t_1), \ldots, Y(t_L)) \) is integrable, then, at a changeable \( T \), its mean value

\[
\bar{m}_\varphi(T) = \mathbb{E}_T \{ \varphi(Y(t_1), \ldots, Y(t_L)) \} = \frac{1}{T/2} \int_{-T/2}^{T/2} \varphi(Y(t_1 + t), \ldots, Y(t_L + t)) dt
\]

represents a hyperrandom function, which may be interpreted as a sequence of hyperrandom quantities.

A hyperrandom function \( X(t) \), stationary under all conditions, will be called ergodic if at all \( g \in \mathbb{G} \) and \( T \to \infty \) the mean value \( \bar{m}_\varphi(T) \) of the function \( \varphi(Y(t_1), \ldots, Y(t_L)) \) converges, almost without exception, to the mean value \( m_\varphi = \mathbb{E}[\varphi(Y(t_1), \ldots, Y(t_L))] \):

\[
\lim_{T \to \infty} \bar{m}_\varphi(T) = m_\varphi.
\]

In other words, it can be expanded in the series of ergodic random functions \( X_g(t) \):

\[
X(t) = \lim_{T \to \infty} \sum_{g} X_g(t - T(g + 0.5)).
\]

In this case the mean value \( \mathbb{E}_T \{ \varphi(Y(t_1), \ldots, Y(t_L)) \} \) of the function \( \varphi(Y(t_1), \ldots, Y(t_L)) \), calculated by an arbitrarily chosen realization \( x(t) \) of the hyperrandom process \( X(t) \), based on averaging over time \( t \), gives (with the probability equal to unity) the mean value, calculated for the function \( \varphi(Y(t_1), \ldots, Y(t_L)) \) based on averaging of a set of realizations of the considered process \( X(t) \).

When the conditions are constant, the hyperrandom process degenerates in a random one. Then the expressions (2)–(3) permit to formulate the widely used definition of the stationary ergodic random process [12].

Assign the name of boundaries of the mean value of the realization \( x(t) \), corresponding to an ergodic hyperrandom function \( X(t) \), to the following functions:

\[
\bar{m}_{x_T} = \sup_{g \in \mathbb{G}} T \int_{T_g} x_g(t) dt,
\]

\[
\underline{m}_{x_T} = \inf_{g \in \mathbb{G}} T \int_{T_g} x_g(t) dt,
\]

the name of boundaries of the covariance function of realization — to the function

\[
\bar{K}_{x_T} (\tau) = \sup_{g \in \mathbb{G}} T \int_{T_g} x_g(t + \tau) x_g(t) dt,
\]

\[
\underline{K}_{x_T} (\tau) = \inf_{g \in \mathbb{G}} T \int_{T_g} x_g(t + \tau) x_g(t) dt.
\]
and the name of boundaries of the correlation function of realization — to the function

\[ \overline{R}_{\xi} (\tau) = \sup_{\xi} \frac{1}{T} \int_{\tau}^{\tau + T} \{ x_{\xi} (t + \tau) - \overline{x}_{\xi} \} [x_{\xi} (t) - \overline{x}_{\xi}] \, dt, \]

\[ \underline{R}_{\xi} (\tau) = \inf_{\xi} \frac{1}{T} \int_{\tau}^{\tau + T} \{ x_{\xi} (t + \tau) - \overline{x}_{\xi} \} [x_{\xi} (t) - \overline{x}_{\xi}] \, dt, \]

where \( \overline{x}_{\xi} = \frac{1}{T} \int_{\tau}^{\tau + T} x_{\xi} (t) \, dt. \)

At \( T \to \infty \) the boundaries of the mean value of the realization \( \overline{x}_{\xi} \) and \( \overline{x}_{\eta} \) will coincide, almost without exception, with the mean value boundaries: \( \overline{x}_{\xi} = \overline{x}_{\eta} \) and \( \overline{x}_{\eta} = \overline{x}_{\delta} \); the boundaries of the covariance function of realization \( \overline{K}_{\xi} (\tau) \) and \( \overline{K}_{\eta} (\tau) \) coincide, almost without exception, with the boundaries of the covariance function \( \overline{K}_{\xi} (\tau) = \overline{K}_{\eta} (\tau) \) and \( \overline{K}_{\eta} (\tau) \), the boundaries of the correlation function of realization \( \overline{R}_{\xi} (\tau) \) and \( \overline{R}_{\eta} (\tau) \) — with the boundaries of the correlation function \( \overline{R}_{\xi} (\tau) \) and \( \overline{R}_{\eta} (\tau) \); and the upper and lower boundary of variance of realization \( \overline{D}_{\xi} = \overline{D}_{\eta} = \overline{D}_{\delta} \) coincide, almost without exception, with the boundaries of the variance \( D_{\xi} = D_{\eta} = D_{\delta} \).

As in the probability theory, for determination of the ergodic hyperrandom function \( X(t) \) we can use some other type of convergence of boundaries of the function averaged values: for example, instead of convergence almost without exception — the convergence in the mean-value sense.

The above results permit extension to the many-dimensional case. Particularly, for the boundaries of the mutual covariance function of the realizations \( x(t) \) and \( y(t) \) of ergodic hyperrandom functions \( X(t) \) and \( Y(t) \) we can take the functions

\[ \overline{K}_{x,y} (\tau) = \sup_{\xi} \frac{1}{T} \int_{\tau}^{\tau + T} x_{\xi} (t + \tau) y_{\xi} (t) \, dt, \]

\[ \overline{K}_{x,y} (\tau) = \inf_{\xi} \frac{1}{T} \int_{\tau}^{\tau + T} x_{\xi} (t + \tau) y_{\xi} (t) \, dt, \]

and for the boundaries of the cross-correlation function of these realizations — the functions

\[ \overline{R}_{x,y} (\tau) = \sup_{\xi} \frac{1}{T} \int_{\tau}^{\tau + T} \{ x_{\xi} (t + \tau) - \overline{x}_{\xi} \} [y_{\xi} (t) - \overline{y}_{\xi}] \, dt, \]

\[ \overline{R}_{x,y} (\tau) = \inf_{\xi} \frac{1}{T} \int_{\tau}^{\tau + T} \{ x_{\xi} (t + \tau) - \overline{x}_{\xi} \} [y_{\xi} (t) - \overline{y}_{\xi}] \, dt. \]

At \( T \to \infty \) the boundaries of the cross-correlation function of realizations \( \overline{K}_{x,y} (\tau) \) and \( \overline{K}_{y,x} (\tau) \) coincide almost without exception with the boundaries of the mutual covariance function \( K_{x,y} (\tau) \) and \( K_{y,x} (\tau) \), while the boundaries of the cross-correlation function of the realizations \( \overline{R}_{x,y} (\tau) \) and \( \overline{R}_{y,x} (\tau) \) coincide almost without exception with the boundaries of the cross-correlation function \( R_{x,y} (\tau) \) and \( R_{y,x} (\tau) \) Acting in the same manner, we can determine other averaged characteristics.

**Application of the theory.** Let us take a concrete example to gain an impression of advantages of practical application of the above methods for treatment of radio-engineering problems.

Consider the oscillations \( X(t) \) and \( Y(t) \), recorded in the ocean by two hydroacoustic receivers located near the surface at a small distance from each other. A noisy object is passing with the velocity \( v \) by the receivers and creates the signals \( S_x(t) \) and \( S_y(t) \) in the reception points. Apart from the above signals, the oscillations \( X(t) \) and \( Y(t) \) contain additive interference components.
The distance between the receivers is small. Therefore, the signals $S_x(t)$ and $S_y(t)$ differ from each other by a delay $\tau(t)$ which is variable in time as the object is moving: $S_y(t) = S_x(t - \tau(t))$. Since the receivers are located near the surface, the interference takes place, so that during object's movement the level of the signals is changing. Taking into account that the surface layer of the ocean is inhomogeneous and the type of this inhomogeneity is usually unknown, the signals $S_x(t)$ and $S_y(t)$ must be regarded as hyperrandom processes. Note that only the signal components but also the interference ones may be of hyperrandom nature. Even if not so, the oscillations $X(t)$ and $Y(t)$ turn out to be hyperrandom processes.

It is difficult to say whether these processes are stationary and ergodic. As a rule, we hardly can prove strictly or reject the hypotheses of stationarity and, especially, of ergodicity. To do this, we must have a large number of realizations obtained under the same totality of conditions. However, with a relatively small amount of experimental data corresponding to a single realization, we cannot make any correct conclusions. In this case, as in many others, these hypotheses can be adopted based on a physical understanding of the mechanism of formation of the processes. Thus, with the use of a single realization, we can calculate various characteristics unifying the processes under investigation.

Figures 1 and 2 give an insight into correlation and spectral characteristics corresponding to a weak legitimate signal approaching to a distinguishable one.

Figure 1a shows estimates of the correlation functions $R_x^g(\tau / g)$ of fragments of the process $X(t)$ corresponding to different conditions $g$ — the position of the moving object in space (thin lines merged into the gray background), the estimate of the upper bound of the correlation function $R_x^u(\tau)$ (the upper bold curve) and the estimate of the lower bound of the correlation function $R_x^l(\tau)$ (the lower bold curve) of the hyperrandom process $X(t)$, and also the estimate of the correlation function $R_x^u(\tau)$ calculated under the assumption that the conditions are constant (a thin dark line between the two bold lines).

Figure 1b shows estimates of the cross-correlation functions $R_y^g(\tau / g)$ of fragments of $X(t)$ and $Y(t)$ corresponding to different conditions $g$ (thin lines merged into a gray background), the estimate of the upper bound of the cross-correlation function $R_y^u(\tau)$ (the upper bold curve) and the estimate of the lower bound of the correlation function $R_y^l(\tau)$ (the lower bold curve) of the processes $X(t)$ and $Y(t)$, and also the estimate of the cross-correlation function $R_y^u(\tau)$ calculated under the assumption that the conditions are constant (the thin dark line).

Figure 2a shows estimates of the upper and lower boundaries of noise radiation spectrum $S_{x0}(f)$, $S_{x1}(f)$ of the process $X(t)$ (the upper and the lower bold curves, respectively), and the estimate of noise radiation spectrum $S_y^*(f)$ calculated under the assumption that the conditions are constant (the thin dark line). In Fig. 2b we can see the estimates of the upper and lower boundaries of noise radiation mutual spectrum $S_{y0}(f)$, $S_{y1}(f)$ of the processes $X(t)$ and $Y(t)$ (the upper and the lower bold curves, respectively), and the estimate of noise radiation mutual spectrum $S_{y0}^*(f)$ calculated under the assumption that the conditions are constant (the thin dark line). The dotted lines are thresholds corresponding to the estimate of the mean value plus the estimate $3\sigma$.

The legitimate signal included a weakly correlated noise component, similar to the hyperrandom white noise in its properties, and discrete components, forming the sound scale, while the interference component approached the random
white noise. The calculation was performed over 100 fragments of the realization, and each fragment contained 512 time samples.

As can be seen from the pictures, the main lobe of the estimate of the upper bound of the correlation function $R_u^*(\tau)$ is wider, while the respective lobe of the estimate of the lower bound of the correlation function $R_l^*(\tau)$ is narrower than the main lobe of the estimate of the correlation function $R_l^*(\tau)$ calculated under the assumption that the conditions are constant. Moreover, the mean arithmetic of the openings of lobes of boundary estimates $R_u^*(\tau)$ and $R_l^*(\tau)$ is roughly equal to the opening of the lobe of correlation function $R_l^*(\tau)$.

The estimate of the upper bound of the cross-correlation function $R_{xy}^*(\tau)$ establishes the time delay between the signals $S_x(t)$ and $S_y(t)$ (see the first large peak in the domain close to zero). True enough, the lower bound estimate $R_{xy}^*(\tau)$ bears here little information, and the cross-correlation function $R_{xy}^*(\tau)$ bears no information at all. This situation arises because the signal level falls rapidly when the object is moving away. With the aid of the upper boundary of the cross-correlation function we can fix delays in several fragments close to each other, where the signals are rather intensive. Weak signals are not perceived by this function. The estimate of the cross-correlation function $R_{xy}^*(\tau)$ is formed by averaging all fragments, including those (constituting the major portion) with the signal level considerably lower than the interference level. Because of this, the graph of $R_{xy}^*(\tau)$ has no large spikes.

The estimates of the upper bound of noise radiation spectrum $S_{nu}^*(f)$ and of the upper bound of mutual noise radiation spectrum $S_{nu}^*(f)$ bear much more information than the estimates of noise radiation spectrum $S_u^*(f)$ and of mutual noise radiation spectrum $S_{xy}^*(f)$ calculated under the assumption that the conditions are constant. They also bear more information than the estimates of the lower bound of the spectrum $S_l^*(f)$ and lower bound of the mutual spectrum $S_{xy}^*(f)$. On the curves $S_{nu}^*(f)$ and $S_{xy}^*(f)$ we can distinctly see the spectral components forming two sound scales with fundamental frequencies 3.2 Hz and 4 Hz. On the curves $S_u^*(f)$, $S_{xy}^*(f)$ and $S_l^*(f)$, $S_{xy}^*(f)$ these components are not seen.

In this case, the spectral processing algorithm, taking the hyperrandom nature of the process into account, gives better results than the spectral algorithm, which assumes that the process is random. A more detailed study of the issue shows that

(a) the gain in the noise stability depends nonlinearly on the signal level;
(b) at minimum permissible signal levels the gain is roughly equal to 6 dB; and
(c) as the signal level increases, the gain falls to nil.

Obviously, two latter assertions refer only to the specific example. In other cases the situation may be different. However, the gain may take place, and be rather essential.

When dealing with a number of radio-engineering problems, one may encounter a situation, when, under the influence of many factors, the observation conditions are varying in an unpredictable manner. Neglecting the fact of variable conditions, or any attempt to describe the dynamics of these changes by the probabilistic methods, as can be seen
from the above example, may lead to wrong results. In the events when the actual processes are obviously hyperrandom, application of methods, taking the hyperrandom properties into account, is much more effective than the probabilistic methods.

**Conclusion.** We formulated the notions in concern with hyperrandom phenomena: stationarity and ergodicity of a hyperrandom function, hyperrandom white noise, etc. We suggested several characteristics to describe the stationary and ergodic hyperrandom functions. We investigated properties of these characteristics and developed spectral methods for description of stationary hyperrandom functions. As a rule, based on experimental data it is hardly possible to prove or reject the hypotheses of stationarity or ergodicity of hyperrandom processes. However, these hypotheses can often be adopted based on physical understanding of the mechanism of the process. We also show that when treating radio-engineering problems related to description of phenomena under uncontrollably changing conditions, the correct account of hyperrandom nature of processes may give a noticeable advantage at minimum permissible levels of legitimate signals.

**REFERENCES**


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